Random variables and expectation: in principle

1. Random variables:
   (a) What is a ***random variable*** on a sample space Ω?
   (b) If $A \subset \Omega$, what is the ***indicator variable*** $\mathbb{1}_A$? How are indicator variables useful?
   (c) If X and Y are random variables on Ω and $c \in \mathbb{R}$, define each of the following:
   (i) cX (ii) $X + Y$ (iii) $X \cdot Y$ (iv) $\min(X, Y)$ (v) $\max(X, Y)$
   (d) What do we mean by the *probability distribution* of a random variable?

2. If X is a random variable on a probability space $(\Omega, \mathbb{P})$, how do we define its ***expected value*** $E(X)$? What properties does expected value have?

. . . and in practice

3. If two fair six-sided dice are rolled, let X be the value of the first roll and Y the value of the second.
   (a) For each combination of the random variables X and Y below, find the distribution for its values and compute its expected value:
   (i) $X + Y$ (ii) $X - Y$ (iii) $\min(X, Y)$ (iv) $\max(X, Y)$
   (b) Determine the distribution for the sum of the random variables in subparts (i) and (ii) above, and briefly explain.
   (c) Determine the distribution for the sum of the random variables in subparts (iii) and (iv) above, and briefly explain.

4. Consider a pair of fair six-sided dice having the following numbers on their faces: Die X: $1, 2, 2, 3, 3, 4$
   Die Y: $1, 3, 4, 5, 6, 8$
   For the event space of rolling both dice (as in the above problem), find the probability distributions for the values X, Y, and $X + Y$; compare with your answer to 3(a)(i).

5. Suppose that a fair coin is flipped 9 times.
   (a) What is the probability of event A_{123}, that the first three flips all have the same result?
   (b) What is the probability of event A_{234}, that the second, third, and fourth flips all have the same result?
   (c) What is the probability $\mathbb{P}(A_{123} \mid A_{234})$? Are these two events independent?
   (d) Represent each of the events $A_{123}, A_{234}, \ldots, A_{789}$ by an indicator variable; what is the expected value of each of these?
   (e) What is the *expected* count of subsequences of three consecutive flips being identical?*

6. Suppose that a permutation of the list $(1, 2, 3, \ldots, n)$ is chosen at random (with each of the $n!$ permutations having equal probability).
   (a) What is the probability that the permutation's first number listed is 1?
   What is the probability that the permutation's second number listed is 2?
   Are these two events independent?
   (b) In general, what is the probability that the permutation's k^{th} number is k?
   Represent each of the above by an indicator variable; what is the expected value of each of these functions?
   (c) What is the *expected* count of numbers k that end up in the k^{th} spot?*

* Hint: Add up the indicator variables, and remember that expectation is additive!